A Simple Algorithm for Coloring m-Clique Holes

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***Abstract*—An m-clique hole is a sequence $\phi = (\Phi_1, \Phi_2, \ldots, \Phi_m)$ of m distinct cliques such that $|\Phi_i| \le m$ for all $i = 1, 2, \ldots, m$, and whose clique graph is a hole on m vertices. That is, ϕ is an m-clique hole if for all $i \neq j$, $i, j = 1, 2, \ldots, m$, $\Phi_i \cap \Phi_j \neq \emptyset$ if and only if $(j-1)$ mod $m = (j+1)$ mod $m = i$ mod m. This paper derives a sufficient and necessary condition on m-colorability of m-clique holes, and proposes a coloring algorithm that colors m-clique holes with exactly m colors.**

***Index Terms*—Imperfect graphs, odd holes, coloring.**

I. Introduction

The intersection graph of a family of non-empty sets is the graph whose vertex set is the sets of the family, and whose edge set is all unordered pairs of vertices whose corresponding sets in the family intersect. A clique in a graph is a complete subgraph maximal under inclusion. The clique graph K(G) of a graph G is the intersection graph of the cliques of G.

A hole is a chordless cycle on at least four vertices. A hole is odd if it has an odd number of vertices. Consider a sequence $\phi = (\Phi_1, \Phi_2, \ldots, \Phi_m)$ of m distinct cliques such that $|\Phi_i| \le m$ for all $i = 1, 2, \ldots, m$. We call the sequence ϕ an *m-clique hole* if the clique graph K(ϕ) is a hole on m vertices; i.e., if for all $i \neq j$, $i, j = 1, 2, \ldots, m$, $\Phi_i \cap \Phi_j \neq \emptyset$ if and only if $(j-1)$ mod $m = (j+1)$ mod $m = i$ mod m. We call ϕ an odd m-clique hole if K(ϕ) is an odd hole. Otherwise, ϕ is an even m-clique hole.

An m-clique hole ϕ is said to be m-colorable if one can color all the vertices in ϕ with m different colors such that no two vertices in ϕ sharing an edge between them are colored with the same color. An odd m-clique hole is imperfect [2], and hence its chromatic number is greater than its clique number m for some of its induced subgraphs [1]; i.e., odd m-clique holes are not m-colorable in general. In this paper, we first provide and prove a sufficient and necessary condition on m-colorability of m-clique holes, and then provide a coloring algorithm that colors m-clique holes with exactly m different colors.

II. A Sufficient and Necessary Condition

In this section, we present and prove the main theorem of the paper which states a sufficient and necessary condition on the colorability of m-clique holes. The proof will also be the basis for the proposed coloring algorithm that we present in Section III. As we go through the proof, we will introduce and prove several lemmas and propositions that will be used for proving the main theorem as well as for designing the proposed coloring algorithm. The proof will go until the end of this section. Throughout this paper, for simplicity of notation, all indices wrap around after reaching m; e.g., Φ_{m+1} refers to Φ_1, Φ_{m+2} refers to Φ_2, etc., and Φ_0 refers to Φ_m, Φ_{-1} refers to Φ_{m-1}, etc. Now, we state the main result.

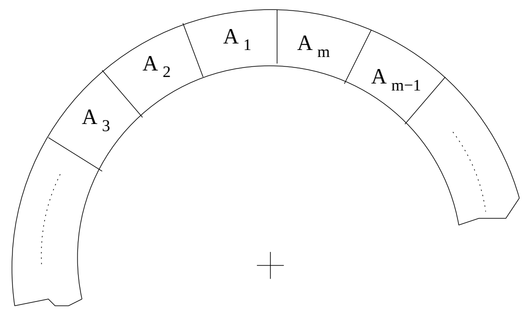

Fig. 1. Ring representation of $\cup_{i=1}^{m} \mathcal{A}_i$.

Theorem *1:* An m-clique hole $\phi = (\Phi_1, \Phi_2, \ldots, \Phi_m)$ is m-colorable if and only if $\sum_{i=1}^{m} |\Phi_i \cap \Phi_{i+1}| \le m\lfloor \frac{m}{2} \rfloor$.

A. The "if" part proof

Let us denote the set of vertices $\Phi_i \cap \Phi_{i+1}$ by $\mathcal{A}_i$ for all $i = 1, 2, \ldots, m$. Suppose that $|\Phi_i| \le m$ for all $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} |\mathcal{A}_i| \le m\lfloor \frac{m}{2} \rfloor$. In this section, we will show that $\phi = (\Phi_1, \Phi_2, \ldots, \Phi_m)$ has a proper m-coloring by proposing an algorithm that colors ϕ with exactly m different colors.

Note that any vertex in $\Phi_i - \{\mathcal{A}_{i-1} \cup \mathcal{A}_i\}$ shares edges with and only with vertices in $\mathcal{A}_{i-1} \cup \mathcal{A}_i$. Now since $|\Phi_i| \le m$ and $\{\mathcal{A}_{i-1} \cup \mathcal{A}_i\} \subseteq \Phi_i$, then the vertices in $\Phi_i - \{\mathcal{A}_{i-1} \cup \mathcal{A}_i\}$ can clearly be m-colored provided that the vertices in $\mathcal{A}_{i-1} \cup \mathcal{A}_i$ already have a proper m-coloring. Therefore, in order for us to color ϕ with m colors, it suffices to provide the vertices in $\cup_{i=1}^{m} \mathcal{A}_i$ with a proper m-coloring. In what follows, we will be concerned only with coloring the vertices in $\cup_{i=1}^{m} \mathcal{A}_i$.

For convenience, let us represent the induced graph from ϕ consisting of only vertices in $\cup_{i=1}^{m} \mathcal{A}_i$ as a ring φ as shown in Fig. 1. In this representation, each sector of the ring corresponds to a set $\mathcal{A}_i$. Observe that each vertex in $\mathcal{A}_i$ shares an edge with and only with every vertex in its own sector $\mathcal{A}_i$ and its two immediate adjacent sectors $\mathcal{A}_{i-1}$ and $\mathcal{A}_{i+1}$. Also, note that

$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$, since otherwise $\Phi_{i'} \cap \Phi_{j'} \neq \emptyset$ holds for some $j' \neq i' + 1$; which contradicts the definition of m-clique holes.

Let us restate the conditions of the theorem to fit into the ring representation. Since for all $i = 1, 2, \ldots, m$, $|\Phi_i| \leq m$, $\mathcal{A}_{i-1} \cup \mathcal{A}_i \subseteq \Phi_i$, and $\mathcal{A}_{i-1} \cap \mathcal{A}_i = \emptyset$, then it follows that $|\mathcal{A}_{i-1}| + |\mathcal{A}_i| \leq m$. Also, since $\Phi_i \cap \Phi_{i+1} \neq \emptyset$, then $|\mathcal{A}_i| \geq 1$ must hold for all $i = 1, 2, \ldots, m$. Finally, we have $\sum_{i=1}^{m} |\mathcal{A}_i| \leq m\lfloor \frac{m}{2} \rfloor$ (stated by theorem condition). Our task is then to provide a proper m-coloring to all the vertices in the ring under these stated conditions. We will consider and prove for the extreme cases of rings whose $\sum_{i=1}^{m} |\mathcal{A}_i|$ is equal to $m\lfloor \frac{m}{2} \rfloor$. It is clear that any other instance for which $\sum_{i=1}^{m} |\mathcal{A}_i| < m\lfloor \frac{m}{2} \rfloor$ can also be m-colored once we provide a proper m-coloring for the extreme cases. Hence, from now on, a ring will refer to one of these extreme instances, and is formally defined as follows.

Definition *1:* A ring φ is a sequence $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ of m cliques such that, for all $i, j = 1, 2, \ldots, m$,

1) $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$;
2) $\mathcal{A}_i \cup \mathcal{A}_{i+1}$ is a maximum clique;
3) $|\mathcal{A}_i| \geq 1$;
4) $|\mathcal{A}_i| + |\mathcal{A}_{i+1}| \leq m$;
5) $\sum_{i=1}^{m} |\mathcal{A}_i| = m\lfloor \frac{m}{2} \rfloor$.

Definition *2:* A ring $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ is balanced (and hence is called a balanced ring) when all sets $\mathcal{A}_i$ are of the same cardinality; i.e., $|\mathcal{A}_i| = \lfloor \frac{m}{2} \rfloor$ for $i = 1, 2, \ldots, m$. A ring is called unbalanced when it is not balanced.

Lemma *1:* A ring $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ is unbalanced if and only if for some $i = 1, 2, \ldots, m$, $|\mathcal{A}_i| < \lfloor \frac{m}{2} \rfloor$ and $|\mathcal{A}_{i-1}| + |\mathcal{A}_i| \leq 2\lfloor \frac{m}{2} \rfloor$.

Proof: The "if" part follows from the definition of the balanced ring. Now we show the "only if" part. Let $\lfloor \frac{m}{2} \rfloor = n$. Since φ is not balanced, there must exist at least one i for which $|\mathcal{A}_i| < n$. Let $\mathcal{B} = \{i : |\mathcal{A}_i| < n, i = 1, 2, \ldots, m\}$ and $k = |\mathcal{B}| > 0$. We now show that for some $i \in \mathcal{B}$, $|\mathcal{A}_{i-1}| + |\mathcal{A}_i| \leq 2n$. Suppose that for all $i \in \mathcal{B}$, $|\mathcal{A}_{i-1}| + |\mathcal{A}_i| > 2n$. Let $\mathcal{B}' = \{i - 1 : i \in \mathcal{B}\}$ and $\bar{\mathcal{B}} = \{1, 2, \ldots, m\} - \{\mathcal{B} \cup \mathcal{B}'\}$. Note that for all $i \in \mathcal{B}'$, $|\mathcal{A}_i| > n$ (i.e., $i \notin \mathcal{B}$) since $i + 1 \in \mathcal{B}$ and thus $|\mathcal{A}_{i+1}| < n$ and $|\mathcal{A}_i| + |\mathcal{A}_{i+1}| > 2n$. Hence, $\mathcal{B}$, $\mathcal{B}'$ and $\bar{\mathcal{B}}$ are disjoint. Also, note that for all $i \in \bar{\mathcal{B}}$, $|\mathcal{A}_i| > n$. Therefore, one can write

$$\begin{aligned}
\sum_{i=1}^{m} |\mathcal{A}_i| &= \sum_{i \in \bar{\mathcal{B}}} |\mathcal{A}_i| + \sum_{i \in \mathcal{B}} |\mathcal{A}_i| + \sum_{i \in \mathcal{B}'} |\mathcal{A}_i| \\
&= \sum_{i \in \bar{\mathcal{B}}} |\mathcal{A}_i| + \sum_{i \in \mathcal{B}} (|\mathcal{A}_{i-1}| + |\mathcal{A}_i|) \\
&> |\bar{\mathcal{B}}| \times n + |\mathcal{B}| \times 2n \\
&= (m - 2k) \times n + k \times 2n \\
&= mn \\
&= m\lfloor \frac{m}{2} \rfloor
\end{aligned}$$

which contradicts the fact that φ is a ring. ■

The difficulty of the "if" part of the proof of Theorem 1 is when m is odd. When m is even, the proof is relatively simple. Therefore, we consider the two parity cases of m separately.

1) CASE 1: $m = 2n + 1$ *is odd:*

Definition *3:* We say that a ring $\varphi' = (\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_m)$ is a transformation of a ring $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ if there exists a unique $i \in \{1, 2, \ldots, m\}$ such that $|\mathcal{A}'_i| = |\mathcal{A}_i| - 1$ and either $|\mathcal{A}'_{i+1}| = |\mathcal{A}_{i+1}| + 1$ or $|\mathcal{A}'_{i-1}| = |\mathcal{A}_{i-1}| + 1$. We then write $\varphi' = T_i(\varphi)$.

Lemma *2:* Let $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ and $\varphi' = (\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_m)$. If $\varphi' = T_i(\varphi)$, then $|\mathcal{A}_i| \geq 2$ and either $|\mathcal{A}_{i+1}| + |\mathcal{A}_{i+2}| \leq m - 1$ or $|\mathcal{A}_{i-1}| + |\mathcal{A}_{i-2}| \leq m - 1$.

Proof: If $|\mathcal{A}_i| < 2$, then $|\mathcal{A}'_i| < 1$. Also, if $|\mathcal{A}_{i+1}| + |\mathcal{A}_{i+2}| > m - 1$ and $|\mathcal{A}_{i-1}| + |\mathcal{A}_{i-2}| > m - 1$, then $|\mathcal{A}'_{i+1}| + |\mathcal{A}'_{i+2}| > m$ or $|\mathcal{A}'_{i-1}| + |\mathcal{A}'_{i-2}| > m$. None of the above can be true since φ' is a ring. ■

Lemma *3:* φ' is a transformation of φ iff φ is a transformation of φ'.

Proof: If $\varphi' = T_i(\varphi)$, then it follows that $\varphi = T_{i-1}(\varphi')$ or $\varphi = T_{i+1}(\varphi')$. ■

Proposition *1:* For every unbalanced ring φ, there exists a sequence of a finite number k of rings $(\varphi^1, \varphi^2, \ldots \varphi^k)$ such that φ^1 is a transformation of the balanced ring φ^0, φ^i is a transformation of φ^{i-1} for $i = 2, 3, \ldots, k$, and φ is a transformation of φ^k.

Proof: Let $\varphi^0 = (\mathcal{A}^0_1, \mathcal{A}^0_2, \ldots, \mathcal{A}^0_m)$ be the balanced ring and $\varphi \neq \varphi^0$ be a ring. Instead of showing that φ can be obtained via a finite number of transformations from φ^0, we show the opposite; that is, we show that φ^0 can be obtained through a finite number of transformations from φ. The proof will then follow from LEMMA 3. We transform φ to φ^0 by moving vertices across the sets $\mathcal{A}_i$ until we obtain $\mathcal{A}_i = n$ for all i (which results in φ^0). The procedure takes place in a finite number of iterations, each of which involves a finite number of transformations. The transformation procedure is as follows.

1) Make $\mathcal{B} = \{i : |\mathcal{A}_i| < n, i = 1, 2, \ldots, m\}$,
2) If $\mathcal{B} = \emptyset$ (i.e., ring is balanced), then stop. Else, pick any $i \in \mathcal{B}$ satisfying LEMMA 1,
3) Find any $j > i$ such that $|\mathcal{A}_j| > n$, and for all $j' = i + 1, i + 2, \ldots, j - 1$, $|\mathcal{A}_{j'}| \leq n$ (it is easy to argue that there exists one since φ is not balanced),
4) Move a vertex from $\mathcal{A}_j$ to $\mathcal{A}_i$. This iteration is attainable in $j - i$ transformations. That is, the first vertex to be moved is from the set $\mathcal{A}_j$ to the set $\mathcal{A}_{j-1}$, then from $\mathcal{A}_{j-1}$ to $\mathcal{A}_{j-2}$, …, from $\mathcal{A}_{i+1}$ to $\mathcal{A}_i$. Go back to Iteration 1.

There are two points that are worth mentioning. First, the choice of j imposed through Iteration 3 assures that when the vertex is moving down, the resulted rings are transformations of their previous ones. This is because the property $|\mathcal{A}_i| + |\mathcal{A}_{i+1}| \leq 2n + 1$ is not violated when moving a vertex from $\mathcal{A}_{i+2}$ down to $\mathcal{A}_{i+1}$. This property is also assured for the last move (from $\mathcal{A}_{i+1}$ to $\mathcal{A}_i$) by the choice of i made at Iteration 2. Second, it is important to notice that the above procedure terminates due

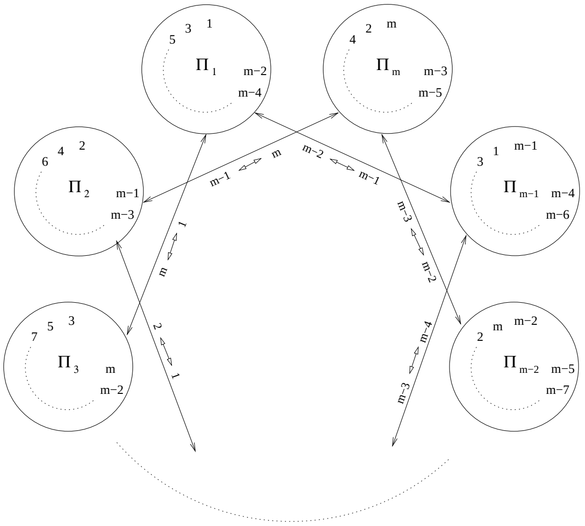

Fig. 2. Coloring Diagram.

to the fact that, by the end of each four iterations, the number $\sum_{i\in\mathcal{B}}(n-|\mathcal{A}_i|)$ decreases by one. Therefore, this number will eventually go to zero resulting in $\mathcal{B}$ being empty; i.e., ring is balanced. ■

Let $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be any given ring. For every $i = 1, 2, \ldots, m$, let Π_i denote the set of all sets that each contains exactly one (any one) vertex of each of the sets $\mathcal{A}_i, \mathcal{A}_{i+2}, \mathcal{A}_{i+4}, \ldots, \mathcal{A}_{m+i-5}, \mathcal{A}_{m+i-3}$. Formally, $\Pi_i = \{\{j_0, j_1, \ldots, j_{n-1}\} \mid j_k \in \mathcal{A}_{i+2k}, k = 0, 1, \ldots, n-1\}$. That is, any set of the form $\{i, i+2, i+4, \ldots, m+i-5, m+i-3\}$ is in Π_i, where an index j in the set refers to any vertex in $\mathcal{A}_j$. Observe that each set in Π_i is an independent set since no two vertices in it present an edge. Because in a cycle of $2n+1$ vertices at most n vertices can be chosen such that no two vertices share an edge between them, an independent set of φ can at most contain n vertices. Hence, for all i, each set in Π_i is a maximum independent set (since each contains n vertices). Fig 2 shows a graphical representation of $\Pi_1, \Pi_2, \ldots, \Pi_m$ as a Diagram. An index j in Π_i refers to any vertex of $\mathcal{A}_j$. An arrow linking two sets Π_i and Π_j indicates that any maximum independent set in Π_i can be transformed to a maximum independent set in Π_j by substituting a vertex for another. Vertices that can be substituted are indicated above the arrow.

Proposition *2:* $\Pi_1, \Pi_2, \ldots, \Pi_m$ are pairwise disjoint.

Proof: Note that each maximum independent set in Π_i is missing the two consecutive vertices $m+i-2$ and $m+i-1$, while exactly one of these two vertices ($m+i-2$ or $m+i-1$) is present in every other maximum independent set in Π_j for $j \neq i$. Hence each set in Π_i does not belong to any set Π_j for $j \neq i$.■

Lemma *4:* Every maximum independent set of any ring belongs to one Π_i for some $i = 1, 2, \ldots, m$.

Proof: Since each maximum independent set must not contain two vertices belonging to two consecutive sets $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ for some i and there are only m possible different pairs of consecutive vertices, then each maximum independent set must not contain one of these m pairs. Hence, every maximum independent set must belong to one of the m sets $\Pi_1, \Pi_2, \ldots, \Pi_m$. ■

Proposition *3:* A ring is balanced if and only if its vertices can be partitioned into exactly m disjoint maximum independent sets $\pi_1^0, \pi_2^0, \ldots, \pi_m^0$ such that $\pi_i^0 \in \Pi_i$ for all $i = 1, 2, \ldots, m$.

Proof: Let $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be the balanced ring; i.e, $|\mathcal{A}_i| = n$ for all i. Let $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$ denote the vertices in $\mathcal{A}_i$ for all i. For each i, we define the set π_i^0 to be $\{v_{i,1}, v_{i+2,2}, v_{i+4,3}, \ldots, v_{m+i-3,n}\}$. For example, $\pi_1^0 = \{v_{1,1}, v_{3,2}, v_{5,3}, \ldots, v_{m-2,n}\}$, $\pi_2^0 = \{v_{2,1}, v_{4,2}, v_{6,3}, \ldots, v_{m-1,n}\}$, and so forth. Clearly, $\pi_i^0 \in \Pi_i$ for all $i = 1, 2, \ldots, m$ and hence $\pi_1^0, \pi_2^0, \ldots, \pi_m^0$ are m disjoint maximum independent sets (follows from PROPOSITION 2). Also, one can easily see that the vertices of each set $\mathcal{A}_i$ are contained in the n distinct sets $\pi_i^0, \pi_{i+2}^0, \ldots, \pi_{m+i-3}^0$ (each vertex is contained in a different set). Hence all the vertices of the ring are partitioned into the m disjoint maximum independent sets. Now suppose that the vertices of a ring can be split into m disjoint maximum independent sets each of which belongs to a different Π_i for some i. Then, all the vertices of each set $\mathcal{A}_i$ are contained in exactly n independent sets, each of which contains no more than one vertex. Hence, $|\mathcal{A}_i| = n$ for all i; i.e., the ring is balanced. ■

Lemma *5:* A ring has exactly m disjoint maximum independent sets if and only if it is m-colorable.

Proof: The forward direction is trivial. Since there are m disjoint sets each of which has n vertices (since they are maximum), then at least mn different vertices can be properly m-colored. This can be done by coloring each independent set with a different color. This results in a proper m-coloring since a ring has mn vertices. Now suppose that a given ring is m-colorable. Then, each color must have been used by at most n vertices because of the fact that in a cycle of $2n+1$ vertices at most n vertices can be chosen such that there is no edge between any two of them. Now since a ring has mn vertices and there are m colors, each color must have been used by exactly n vertices. Therefore, there must be at least m disjoint independent sets of n vertices each. Hence there must be at least m disjoint maximum independent sets (since size of these independent sets is n). Again because a ring has mn vertices, there must be exactly m disjoint maximum independent sets. ■

Lemma *6:* Every set in Π_i for $i = 1, 2, \ldots, m$ can be transformed to a set in Π_{i+2} by substituting i for $i-1$ or to a set in Π_{i-2} by substituting $i-3$ for $i-2$.

Proof: Note that each set in Π_i does not contain a vertex $i-1$, nor a vertex $i-2$, but contains a vertex i. Hence, if i is substituted for $i-1$, then $i-1$ still does share edges with

any of the vertices in the set. This new obtained set is indeed in Π_{i+2}. Similarly, we can prove for the case of Π_{i-2}. ■

Note that the above two transformations are the only two transformations involving the substitution of only one vertex that transform a set in Π_i to another in Π_j's. This is because the insertion of any other vertex will interfere with an already existing vertex. The two-way arrows in the Coloring Diagram provided in Fig. 2 shows all possible substitutions. This diagram will be an essential part of the proposed coloring algorithm that we describe in the next section.

Proposition *4:* Let φ and φ' be two rings that are transformations of one another. φ is m-colorable if and only if φ' is m-colorable.

Proof: We only need to prove one direction; the other follows from LEMMA 3. Suppose $\varphi' = (\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_m)$ is a transformation of $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ and φ is m-colorable. Also, let p be the index such that $|\mathcal{A}'_p| = |\mathcal{A}_p| - 1$ and $|\mathcal{A}'_{p+1}| = |\mathcal{A}_{p+1}| + 1$ (the case where $|\mathcal{A}'_{p-1}| = |\mathcal{A}_{p-1}| + 1$ can be proven similarly). We need to show that φ' is m-colorable. From LEMMA 2, it follows that $|\mathcal{A}_{p+1}| + |\mathcal{A}_{p+2}| \leq m - 1$ must hold. First, observe that one and only one vertex belonging to either one the two successive sets, $\mathcal{A}_{p+1}$ and $\mathcal{A}_{p+2}$, belongs to each set in Π_i, except those in Π_{p+3} which contain none of the two vertices. Second, since φ is m-colorable, then it follows from LEMMA 5 that the vertices of φ are the union of m disjoint maximum independent sets. Now LEMMA 4 implies that each of these m sets must belong to one Π_i for $i = 1, 2, \ldots, m$. Hence, in order for $|\mathcal{A}_{p+1}| + |\mathcal{A}_{p+2}|$ to be less than or equal to $m - 1$, Π_{p+3} must be one of these m disjoint sets. Now using LEMMA 6, one can transform any maximal independent set in Π_{p+3} to the maximal independent set in Π_{p+1} by substituting the vertex p for $p + 1$. Hence, the new m maximum independent sets whose union is φ' are also disjoint. φ' is m-colorable by LEMMA 5. ■

Now we provide the proof of the "if" part of the main theorem (THEOREM 1) when m is odd, which now follows from the derived propositions. First, using PROPOSITION 3, one can properly color the balanced ring with m different colors (each maximal independent set is colored with a different color). Hence the balanced ring is m-colorable. Second, by PROPOSITION 1 we know that any ring can be obtained from the balanced ring through a finite number of transformations. Finally, a ring has a proper m coloring follows from PROPOSITION 4. This ends the proof of sufficiency part of the THEOREM 1 when m is odd. Note that this proof is nothing but a coloring algorithm that colors ϕ with exactly m colors. This algorithm is formally presented in Section III.

2) CASE 2: *$m = 2n$ is even:* When m is even, the proof (and hence a coloring algorithm) is simple; i.e., it does not require rearrangement of colors. Let $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be a ring such that $\sum_{i=1}^{m} \mathcal{A}_i = mn$. The following is a proper m-coloring of φ. Let $\{c_1, c_2, \ldots, c_m\}$ denote the set of m different colors. For each $i = 1, 2, \ldots, m$, color the vertices in $\mathcal{A}_i$ with $\{c_1, c_2, \ldots, c_{|\mathcal{A}_i|}\}$ if i is odd and with $\{c_m, c_{m-1}, \ldots, c_{m-|\mathcal{A}_i|+1}\}$ if i is even. Clearly, this is a proper m-coloring because, for any i, none of the vertices in $\mathcal{A}_i$ share the same color with a vertex in $\mathcal{A}_{i-1} \cup \mathcal{A}_{i+1}$ (note that these vertices are the only ones that share edges with vertices in $\mathcal{A}_i$). This is due to the fact that $|\mathcal{A}_{i-1}| + |\mathcal{A}_i| \leq m$ and $|\mathcal{A}_i| + |\mathcal{A}_{i+1}| \leq m$.

B. The "only if" part proof

Let $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be an m-colorable ring. Since φ is m-colorable, then there must exist at most m disjoint independent sets whose union is φ. Now because each independent set cannot contain more than $\lfloor \frac{m}{2} \rfloor$ vertices (due to cycles of length m), then at most the ring contains $m\lfloor \frac{m}{2} \rfloor$ vertices.

III. A COLORING ALGORITHM

The proposed coloring algorithm that we present in this section follows from the proof presented in Section II. After describing the algorithm for the general case, we illustrate it through an example with $m = 7$. Let us consider a ring $\varphi = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ and let $m = 2n+1$. The key idea of the proposed coloring algorithm is to partition the vertices of φ into m disjoint maximum independent sets—hence, coloring each of them with a different color yields to a proper m-coloring of φ. In Section II, we showed that φ is indeed partitionable into m disjoint maximum independent sets, each of which must belong to Π_i for some $i = 1, 2, \ldots, m$. Let s_i denote the number of maximum independent sets among these m sets which belong to Π_i (i.e., $\sum_{i=1}^{m} s_i = m$). The coloring algorithm consists then of determining s_i for all i. We now describe the different steps of the algorithm.

1) $s_i \leftarrow 1$ for all $i = 1, 2, \ldots, m$,
2) $\mathcal{B} \leftarrow \{i : |\mathcal{A}_i| < n, i = 1, 2, \ldots, m\}$,
3) While $\mathcal{B} \neq \emptyset$, do
    a) Pick any $i \in \mathcal{B}$ such that $|\mathcal{A}_{i-1}| + |\mathcal{A}_i| \leq 2n$,
    b) Pick any $j > i$ such that $|\mathcal{A}_j| > n$ and $|\mathcal{A}_{j'}| \leq n$ for all $j' = i + 1, i + 2, \ldots, j - 1$,
    c) $s_k \leftarrow s_k - 1$ for $k = j + 2, j + 1$,
    d) $s_k \leftarrow s_k + 1$ for $k = i + 2, i + 1$,
    e) $|\mathcal{A}_j| \leftarrow |\mathcal{A}_j| - 1$,
    f) $|\mathcal{A}_i| \leftarrow |\mathcal{A}_i| + 1$,
    g) $\mathcal{B} \leftarrow \{i : |\mathcal{A}_i| < n, i = 1, 2, \ldots, m\}$.

Although the above coloring algorithm follows straightly from the proof the theorem provided in Section II, it is worth bringing the attention to the following three points. First, note that instead of transforming the balanced ring to the ring in question, we proceed in the opposite direction; that is, we apply a finite number of transformations to the unbalanced ring until the balanced ring is obtained. Second, Step 1) follows from the fact that $s_i = 1$ for all $i = 1, 2, \ldots, m$ when the ring is balanced. Third, Steps 3c) and 3d) follow from LEMMA 6 which states that moving a vertex from $\mathcal{A}_k$ to $\mathcal{A}_{k-1}$ is equivalent to substituting a maximum independent set in Π_k

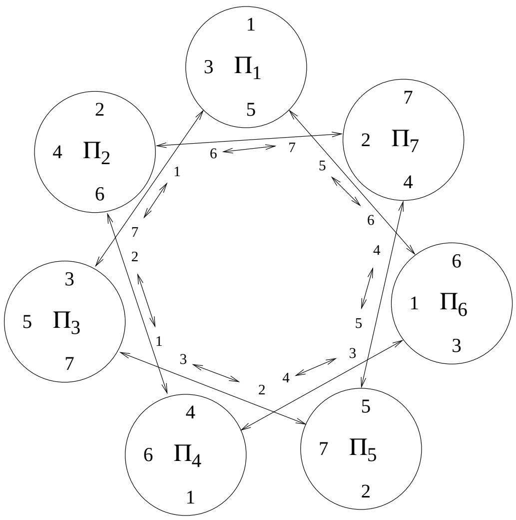

Fig. 3. Coloring Diagram for $m = 7$.

TABLE I
COLORING ALGORITHM STEPS FOR THE EXAMPLE

Iterations	initial	Iter. 1	Iter. 2	Iter. 3	Iter. 4
$\lvert\mathcal{A}_1\rvert$	5	4	4	4	3
$\lvert\mathcal{A}_2\rvert$	2	2	3	3	3
$\lvert\mathcal{A}_3\rvert$	3	3	3	3	3
$\lvert\mathcal{A}_4\rvert$	4	4	3	3	3
$\lvert\mathcal{A}_5\rvert$	1	1	1	2	3
$\lvert\mathcal{A}_6\rvert$	4	4	4	3	3
$\lvert\mathcal{A}_7\rvert$	2	3	3	3	3
s_1	1	2	2	1	1
s_2	1	1	1	1	0
s_3	1	0	1	1	0
s_4	1	1	2	2	2
s_5	1	1	0	0	0
s_6	1	1	0	1	2
s_7	1	1	1	1	2
$\mathcal{B}$	$\{2,5,7\}$	$\{2,5\}$	$\{5\}$	$\{5\}$	$\emptyset$
i	7	2	5	5	–
j	1	4	6	1	–

for one in Π_{k+2} for all $k = j, j-1, \ldots, i+1$ (the in the lemma, we prove that such a substitution exists).

EXAMPLE. Let us now apply the coloring algorithm to an example. Consider the case when $m = 7$ (i.e., $n = 3$). The coloring diagram for $m = 7$ is shown in Fig. 3. Let φ be the example ring $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_7)$ with $|\mathcal{A}_1| = 5$, $|\mathcal{A}_2| = 2$, $|\mathcal{A}_3| = 3$, $|\mathcal{A}_4| = 4$, $|\mathcal{A}_5| = 1$, $|\mathcal{A}_6| = 4$, and $|\mathcal{A}_7| = 2$. All the steps of the coloring algorithm are executed and shown in Table I. When the algorithm terminates, the m disjoint maximum independent sets of φ are all provided through the numbers s_i for all $i = 1, 2, \ldots, m$, which are the outcome of the algorithm. As shown in Table I, these numbers are $s_2 = s_3 = s_5 = 0$, $s_1 = 1$, and $s_4 = s_6 = s_7 = 2$. Therefore, the vertices of φ can be partitioned into the following 7 disjoint maximum independent sets: one set of the form $\{1, 3, 5\}$ ($s_1 = 1$); two sets each of the form $\{4, 6, 1\}$ ($s_4 = 2$); two sets each of the form $\{6, 1, 3\}$ ($s_6 = 2$); and two sets each of the form $\{7, 2, 4\}$ ($s_7 = 2$). This yields to $|\mathcal{A}_1| = 5$, $|\mathcal{A}_2| = 2$, $|\mathcal{A}_3| = 3$, $|\mathcal{A}_4| = 4$, $|\mathcal{A}_5| = 1$, $|\mathcal{A}_6| = 4$, and $|\mathcal{A}_7| = 2$. Coloring each maximum independent set of the 7 sets with a different color results in a proper 7-coloring of φ.

IV. CONCLUSION

In this work, we derived and proved a sufficient and necessary condition on m-colorability of odd m-clique holes, and proposed a coloring algorithm that colors m-clique holes with exactly m colors.

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